Quantum Field Theory

Set 7: solutions

Exercise 1

Every irreducible finite-dimensional representation of the Lorentz group is defined by a couple (j_-, j_+) where j_+ and j_- label the irreducible representation of the two commuting $SU(2)$ subgroups of $SO(3,1) \sim SU(2)_+ \otimes SU(2)_-$ generated by

$$\mathbf{J}_\pm = \frac{\mathbf{J} \pm i\mathbf{K}}{2}. \tag{1}$$

Notice that given representations $D_{j_\pm}$ of $SU(2)$ on vector spaces $V_\pm$, the (j_-, j_+) representation act on the vector space $V_- \otimes V_+$ as the tensor product representation of D_{j_-} and D_{j_+}. Both $\mathbf{J}_\pm$ are in particular defined on $V_- \otimes V_+$, as the generators of such representations.

Consider now the $(1/2, 0)$ representation. In this case V_- is 2-dimensional and V_+ is 1-dimensional, with D_{j_+} being the trivial representation. We can thus forget about V_+ in the product $V_- \otimes V_+$ and simply write

$$\mathbf{J}_- = \frac{\vec{\sigma}}{2}, \qquad \mathbf{J}_+ = \vec{0}_{2\times 2} \tag{2}$$

where $\vec{\sigma}$ is the vector of the three Pauli matrices, which furnish the spin-1/2 representation of $SU(2)$. From this the form of $\mathbf{J}$ and $\mathbf{K}$ in the $(1/2, 0)$ follows from eq 1 and 2

$$\mathbf{J} = \frac{\vec{\sigma}}{2}, \qquad \mathbf{K} = i\frac{\vec{\sigma}}{2}. \tag{3}$$

Given a set of parameters $\vec{\alpha}$ for rotations and another one $\vec{\beta}$ for boosts the explicit form of the elements of the group representation is

$$D_{(1/2,0)}(\vec{\alpha}, \vec{\beta}) = e^{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\alpha}+i\vec{\beta})}. \tag{4}$$

$D_{(1/2,0)}$ acts on 2-dimensional complex vector with an index a such as

$$s_a \to [D_{(1/2,0)}(\vec{\alpha}, \vec{\beta})]_a{}^b s_b. \tag{5}$$

Notice in particular that $D_{(1/2,0)}(\vec{\alpha}, \vec{\beta})$ is an invertible linear transformation with unit determinant, so that it belongs to $SL(2,\mathbb{C})$ (the universal covering group of $SO(3,1)$).

With a similar reasoning one obtains the explicit form of the $(0, 1/2)$ representation, which differs in the sign of the boost generator $\mathbf{K}$. Since $D_{(1/2,0)}$ and $D_{(0,1/2)}$ are not equivalent, the latter representation acts on a different set of indices. If we denote such indices as dotted one we have

$$s_{\dot{a}} \to [D_{(0,1/2)}(\vec{\alpha}, \vec{\beta})]_{\dot{a}}{}^{\dot{b}} s_{\dot{b}}. \tag{6}$$

The $(1/2, 1/2)$ representation is now readily constructed. Indeed, it is the direct product of the previous representations. One introduces an object with 2 kinds of indices, a dotted one, transforming in the $(0, 1/2)$ representation and an un-dotted one transforming under the $(1/2, 0)$:

$$v_{a\dot{a}} \to [D_{(1/2,0)}(\vec{\alpha}, \vec{\beta})]_a{}^b [D_{(0,1/2)}(\vec{\alpha}, \vec{\beta})]_{\dot{a}}{}^{\dot{b}} v_{b\dot{b}}. \tag{7}$$

The object v is defined by 4 complex (8 real) parameters.
Note that the representation $D_{(j_+,j_-)}$ is not unitary. This is consistent with the fact that the Lorentz group, being non-compact, does not admit finite-dimensional unitary representations (but it does admit infinite-dimensional unitary representations, which are required to represent physical states).

Exercise 2

The implementation of a group on functions presents some subtleties. Let us review in general how the representation must be implemented. Consider a group $\mathcal{G}$ that acts on spacetime coordinates as follows:

$$\mathcal{G}:\ x \xrightarrow{g_1} g_1(x) \xrightarrow{g_2} g_2(g_1(x)) = g_2 \circ g_1(x) \equiv g_3(x).$$

The action of $\mathcal{G}$ on the functions of spacetime coordinates is defined through the action of the inverse element on coordinates. The reason for this is that we have defined a scalar function to be a map from events to real numbers. A transformation of $\mathcal{G}$ on x^μ is just a relabeling of the same event in a different frame. This means that $\phi'(e) = \phi(e)$, i.e. every event is still mapped to the same number even after the transformation. Writing explicitly the event e in the two different frames gives us:

$$\mathcal{D}_{g_1}:\ \phi(x) \xrightarrow{g_1} \phi'(g_1(x)) = \phi(x) \implies \phi'(x) = \mathcal{D}_{g_1}[\phi](x) = \phi(g_1^{-1}(x)).$$

The correct implementation of the composition of transformations on the space of functions is thus the following:

$$\mathcal{D}_{g_3}:\ \phi(x) \xrightarrow{g_3} \phi'(g_3(x)) \equiv \phi(x) \implies \phi'(x) = \mathcal{D}_{g_3}[\phi](x) = \phi((g_2 \circ g_1)^{-1}(x)).$$

Let us see how this applies to a transformation of the Poincaré group. The action on coordinates is defined as

$$\mathcal{P}:\ x^\mu \xrightarrow{(\Lambda_1, a_1)} (\Lambda_1)^\mu{}_\nu x^\nu + a^\mu \equiv P_{(\Lambda_1,a_1)}(x) \xrightarrow{(\Lambda_2,a_2)} (\Lambda_2)^\mu{}_\nu (\Lambda_1)^\nu{}_\rho x^\rho + (\Lambda_2)^\mu{}_\nu a_1^\nu + a_2^\mu$$
$$P_{(\Lambda_2,a_2)} \circ P_{(\Lambda_1,a_1)}(x) \equiv P_{(\Lambda_3,a_3)}(x).$$

Thus the composition of the transformation $(\Lambda_2,\, a_2)$ and $(\Lambda_1,\, a_1)$ gives a third transformation with parameters $(\Lambda_3, a_3) = (\Lambda_2\Lambda_1,\, \Lambda_2 a_1 + a_2)$.
We now verify that the implementation on functions presented in the text reproduces this composition rule (we suppress spacetime indices for shortness):

$$\mathcal{D}_{(\Lambda_3,a_3)}:\ \phi(x) \xrightarrow{(\Lambda_3,a_3)} \phi'(\Lambda_3 x + a_3) \equiv \phi(x) \implies \phi'(x) = \mathcal{D}_{(\Lambda_3,a_3)}[\phi](x) = \phi(\Lambda_1^{-1}\Lambda_2^{-1}(x - \Lambda_2 a_1 - a_2))$$

and

$$\mathcal{D}_{(\Lambda_2,a_2)}\mathcal{D}_{(\Lambda_1,a_1)}:\ \phi(x) \ \to \phi'(P_{(\Lambda_2,a_2)} \circ P_{(\Lambda_1,a_1)}(x)) \equiv \phi(x)$$
$$\implies \phi'(x) = \mathcal{D}_{(\Lambda_2,a_2)}\mathcal{D}_{(\Lambda_1,a_1)}[\phi](x) = \phi(P^{-1}_{(\Lambda_1,a_1)} \circ P^{-1}_{(\Lambda_2,a_2)}(x)) = \phi(\Lambda_1^{-1}(\Lambda_2^{-1}(x - a_2) - a_1)).$$

Thus, with the rule $\phi'(x) = \phi(\Lambda^{-1}(x - a))$ one gets that the composition of transformations is respected, i.e. acting on functions with $\mathcal{D}_{(\Lambda_3,a_3)}$ or with $\mathcal{D}_{(\Lambda_2,a_2)}\mathcal{D}_{(\Lambda_1,a_1)}$ is the same, as it is acting on fourvectors with $P_{(\Lambda_3,a_3)}$ or with $P_{(\Lambda_2,a_2)} \circ P_{(\Lambda_1,a_1)}$. Moreover, since the identity e corresponds to parameters $(\Lambda = 1_4, a = 0)$, then

$$\mathcal{D}_{(1_4,0)}:\ \phi(x) \ \to \phi'(1_4 x + 0) = \phi'(x) = \phi(x),$$

and so the identity has the correct representation (it does not change the functional form of ϕ), proving that indeed the transformations presented in the text of the exercise define the action of the Poincaré group on functions.

Exercise 3

Consider a field $\phi_a(x)$ which belongs to a given representation M of the Poincaré group. A transformation acting on coordinates as

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu,$$

where Λ is an element of the Lorentz group and a^μ is a spacetime translation, induces a transformation on the field defined as follows:

$$\phi_a(x) \longrightarrow \phi'_a(x') = M(\Lambda)_a{}^b\, \phi_b(x) \implies \phi'_a(x) = M(\Lambda)_a{}^b\, \phi_b(\Lambda^{-1}(x - a)),$$

where the matrix $M(\Lambda)_a{}^b$ is the representative of the Lorentz transformation in the representation ϕ_a belongs to and acts on the index a only. For example:

- In the scalar representation the Lorentz group is trivially represented and $M(\Lambda)_a{}^b = 1$ for all Λ. Therefore

$$\phi'(x) = \phi(\Lambda^{-1}(x-a)).$$

- In the vector representation the field transforms like the coordinate four vector and therefore the representation of the group element is Λ itself:

$$\phi'^{\rho}(x) = \Lambda^{\rho}{}_{\sigma}\,\phi^{\sigma}(\Lambda^{-1}(x-a)).$$

- In the spinorial representation, considered in the previous exercise, the matrix $M(\Lambda)_a{}^b$ is a 2×2 matrix such that, for pure rotations, it coincides with an $SU(2)$ matrix.

Let's consider a general action

$$\mathcal{S} = \int dtd^3x\ \mathcal{L}[\phi](x),$$

and consider the transformation acting on coordinates and fields:

$$\begin{aligned}
&x \longrightarrow x' \equiv f(x), \qquad x = f^{-1}(x'),\\
&\phi(x) \longrightarrow \phi'(x') = M[\phi](x) = M[\phi](f^{-1}(x')), \qquad \phi(x) = M^{-1}[\phi'](f(x)).
\end{aligned}$$

One can then implement the transformation on x as the usual change of coordinates in an integral, and in addition express the field ϕ as a function of the transformed one:

$$\mathcal{S} = \int d^4x\ \mathcal{L}[\phi](x) = \int d^4x'|J|\mathcal{L}[M^{-1}[\phi']](f^{-1}x') \equiv \int d^4x'\mathcal{L}'[\phi'](x').$$

A group of transformation is said to be a symmetry of a theory if the equations of motion have the same structure in terms of transformed quantities. A sufficient condition for this to happen is that the dependence on ϕ' of the functional $\mathcal{L}'$ be exactly the same dependence on ϕ of $\mathcal{L}$. The form of the Lagrangian has to be the same once we express it in terms of transformed field, i.e. $\mathcal{L} = \mathcal{L}'$ as a function, or

$$\text{if} \quad \mathcal{S} \equiv \int d^4x\ \mathcal{L}[\phi](x) = \int d^4x'\mathcal{L}[\phi'](x') \quad \Longrightarrow \quad \text{symmetry.}$$

Notice that in the right hand side of last equation the function is $\mathcal{L}$, not $\mathcal{L}'$ as in the previous equation (if it were $\mathcal{L}'$ then there would be no symmetry, but simply a trivial renaming of quantities). If this is the case the Euler-Lagrange equation of motion will have the same structure in terms of the transformed fields and therefore the dynamics will be unchanged.

One can verify that this is the case for Poincaré transformations in scalar field theory and electromagnetism. The Lagrangian density for a real scalar field reads

$$\mathcal{L}[\phi](x) = \frac{1}{2}\partial_\mu\phi(x)\,\partial^\mu\phi(x) - V[\phi](x).$$

In order to check the invariance of the above Lagrangian density one can write the action in terms of the transformed fields and coordinates and see if the functional form of the Lagrangian is the same as in terms of the untransformed quantities:

$$\begin{aligned}
\mathcal{S} = \int d^4x\mathcal{L}[\phi](x) &= \int d^4x\left(\frac{1}{2}\partial_\mu\phi(x)\,\partial_\rho\phi(x)\eta^{\mu\rho} - V[\phi](x)\right)\\
&= \int d^4x'|J|\left(\frac{1}{2}\partial'_\alpha\phi'(x')\,\partial'_\beta\phi'(x')\eta^{\mu\rho}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\rho} - V[\phi'](x')\right)\\
&= \int d^4x'\left(\frac{1}{2}\partial'_\alpha\phi'(x')\,\partial'_\beta\phi'(x')\eta^{\alpha\beta} - V[\phi'](x')\right)\\
&= \int d^4x'\mathcal{L}[\phi'](x').
\end{aligned}$$

In writing last equations we have used the following properties:

$$\phi'(x') = \phi(x),$$
$$\partial_\mu \phi(x) = \frac{\partial x'^\nu}{\partial x^\mu} \partial'_\nu \phi'(x') = \Lambda^\nu{}_\mu \partial'_\nu \phi'(x'),$$
$$|J| = \det\left(\frac{\partial x^\alpha}{\partial x'^\beta}\right) = \det\left(\Lambda_\beta{}^\alpha\right) = 1 \qquad \text{by definition of } SO(1,3),$$
$$\Lambda^\alpha{}_\mu \Lambda^\beta{}_\rho \eta^{\mu\rho} = \eta^{\alpha\beta}.$$

Therefore the functional dependence of the Lagrangian upon the quantities ϕ and x^μ is the same as the one upon the transformed quantities, i.e. the Lagrangian remains the *same* function after the transformation. The Poincaré group is hence a symmetry of this theory.

One can repeat the argument for the Lagrangian of the electromagnetic field

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

At variance with the scalar field, $A_\nu(x)$ transforms in the vector representation of the Lorentz group, therefore:

$$\begin{aligned} A_\rho(x) &= \Lambda^\mu{}_\rho A'_\mu(x'), \\ \partial_\mu A_\nu(x) &= \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \partial'_\rho A'_\sigma(x'), \\ F_{\mu\nu}(x) &= \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu F'_{\rho\sigma}(x'). \end{aligned}$$

As before, the Lagrangian has the same functional dependence upon the primed quantities as upon the untransformed ones and this implies that the Poincaré group is indeed a symmetry of the theory:

$$\mathcal{S} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)\right) = \int d^4x' \left(-\frac{1}{4} F'_{\mu\nu}(x') F'^{\mu\nu}(x')\right).$$

A note on Lorentz transformations

The Lorentz Group is defined by the matrices satisfying the relation

$$\Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta^{\mu\rho} = \eta^{\nu\sigma},$$

and normally one defines the transformation of a vector with lower index (covariant vector) as $v_\mu \longrightarrow \Lambda_\mu{}^\nu \, v_\nu$. However, introducing the vector (contravariant vector) with upper index as $v^\alpha = \eta^{\alpha\beta} v_\beta$, one obtains the transformation law for this vector as $v^\mu \longrightarrow \Lambda^\mu{}_\nu \, v^\nu$, where by definition

$$\Lambda^\mu{}_\nu \equiv \eta^{\mu\rho} \eta_{\nu\sigma} \Lambda_\rho{}^\sigma,$$

and it can easily shown that it defines a Lorentz transformation as well:

$$\Lambda^\mu{}_\nu \Lambda^\alpha{}_\beta \eta^{\nu\beta} = \eta^{\mu\alpha}.$$

This equation together with the first of this section can be used to express the form of the inverse of a Lorentz transformation:

$$\begin{aligned} &(\Lambda^{-1})_\nu{}^\alpha \eta^{\nu\sigma} = (\Lambda^{-1})_\nu{}^\alpha \Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta^{\mu\rho} = \delta_\mu{}^\alpha \Lambda_\rho{}^\sigma \eta^{\mu\rho} \\ &\Longrightarrow (\Lambda^{-1})_\nu{}^\alpha = \Lambda^\alpha{}_\nu, \end{aligned}$$

and

$$(\Lambda^{-1})^\gamma{}_\mu \eta^{\mu\alpha} = (\Lambda^{-1})^\gamma{}_\mu \Lambda^\mu{}_\nu \Lambda^\alpha{}_\beta \eta^{\nu\beta} = \delta^\gamma{}_\nu \Lambda^\alpha{}_\beta \eta^{\nu\beta} \tag{8}$$
$$\Longrightarrow (\Lambda^{-1})^\gamma{}_\mu = \Lambda_\mu{}^\gamma. \tag{9}$$

Using a matrix notation defining $\Lambda^\mu{}_\nu x^\nu = \Lambda \cdot x$, (where $\Lambda^\mu{}_\nu$ is the element of the μth row and νth column) the statement that Λ is an element of the Lorentz group

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta} \tag{10}$$

can be phrased as

$$\Lambda^T \cdot \eta \cdot \Lambda = \eta \tag{11}$$

where $\eta = \mathrm{diag}(1, -1, -1, -1)$. Multiplying both sides of eq. 11 by Λ^{-1} on the right and η on the left one gets

$$\Lambda^{-1} = \eta \cdot \Lambda^T \cdot \eta \tag{12}$$

where $\eta \cdot \eta = 1$ has been used.